Indian Statistical Institute, Bangalore Centre.
Back-paper Exam : Graph Theory

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Max. points : 60. **Time Limit : 3 hours.**

Answer any three questions only.

Give necessary justifications and explanations for all your arguments. If you are citing results from the class or assignments, mention it clearly. See the end of the question paper for notations.

1. Let G be a simple, undirected graph and $s \neq t \in V(G)$.
   (a) Consider the network with all edge-capacities equal to 1. Show that if f is an integral flow from s to t of strength at least k then there exist k edge disjoint paths from s to t. (**5**).
   (b) State and prove Menger's theorem for edge disjoint paths from s to t in G. (**15**)

2. (a) A plane graph is a k-angulation if every face has length k. Let G be a connected graph and a k-angulation on n vertices and m edges. Show that $m = (n-2)\frac{k}{k-2}$. (**3**)
   (b) Is there a K_5 minor of the Petersen graph ? (**2**)
   (c) Show that the Petersen graph has a $K_{3,3}$ minor. (Hint : Start by deleting one vertex and do not delete any edges.) (**5**).
   (d) Show that the coefficient of x^{n-2} in the chromatic polynomial of a n vertex graph is $\frac{m(m-1)}{2} - T$ where m is the number of edges and T is the number of triangles. (**10**)

3. (a) Compute the eigenvalues of the Laplacian and Adjacency matrices of the the cycle graph C_n. (**10**)
   (b) Find the chromatic polynomial of the cycle graph C_n, wheel graph W_n (i.e., the graph obtained by adding a new vertex to C_n and connecting it to all the n vertices of C_n) and the path graph P_n. (**10**)

4. (a) Let G be an undirected graph with vertex set $\{s, t, 2, \ldots, 7\}$ with the following edge capacities : $c(s,2) = 8, c(s,3) = 12, c(2,5) = 5, c(3,4) = 5, c(3,6) = 4, c(4,5) = 3, c(4,7) = 3, c(5,6) = 9, c(6,7) = 3, c(6,t) = 4, c(7,t) = 10$. We have used $c(u,v)$ to denote capacity for edge (u,v). The edge-capacities which are not specified are assumed to be 0. As usual, s denotes the source and t denotes the target.

(i) Draw the network with the capacities on each edge marked.(**3**)

(ii) Run the Ford-Fulkerson algorithm to find the maximum flow[1]. Draw/mention the flows at each step of the algorithm. (**4**)

(iii) Describe a minimal cut in the above network. Is it unique ? (**3**)

(b) Consider a directed graph $G = (V, E)$ and $s \neq t \in V$. Further assume that there are no incoming edges at s or no out-going edges at t. If f is an integral flow of strength k, show that there exist k directed paths $p_1, \ldots, p_k$ such that for all $e \in E$, $|\{p_i : e \in p_i\}| \leq f(e)$. (**10**)

[1]Here the flow is on the undirected graph G i.e., satisfies anti-symmetry, Kirchoff's node law and non-negative output at the source s